

Single-Letter Characterization of Epsilon-Capacity for Mixed Memoryless Channels

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Abstract—For the class of mixed channels decomposed into stationary memoryless channels, single-letter characterizations of the ε -capacity have not been known except for restricted classes of channels such as the regular decomposable channel introduced by Winkelbauer. This paper gives single-letter characterizations of ε -capacity for mixed channels decomposed into at most countably many memoryless channels with a finite input alphabet and a general output alphabet with/without cost constraints. It is shown that a given characterization reduces to the one for the channel capacity given by Ahlswede when ε is zero. In the proof of the coding theorem, the meta converse bound, originally given by Polyanskiy, Poor and Verdú, is particularized for the mixed channel decomposed into general component channels.

I. INTRODUCTION

The maximum rate of sequence of codes that can attain a decoding error probability less than $\varepsilon \in [0, 1]$ is called the ε -capacity. It is well-known that stationary memoryless channels have the so-called *strong converse property*, and the ε -capacity coincides with the channel capacity (ε -capacity with $\varepsilon = 0$) [16]. On the other hand, allowing a decoding error probability up to ε , the maximum achievable rate may be improved for non-stationary and/or non-ergodic channels. The simplest example is *mixed channels* [5] (also referred to as decomposable channels [15] or averaged channels [1], [7]) whose probability distribution is characterized by a mixture of multiple stationary memoryless channels. This channel is stationary but non-ergodic, and is theoretically important as basic example to be investigated when extensions of coding theorems for ergodic channels are addressed. This channel is known to give the simplest mathematical model of (non-ergodic) block fading channels (c.f. [10], [17]).

For general channels including mixed channels, a general formula of ε -capacity has been given by Verdú and Han [14]. This formula, however, involves limit operations with respect to the code length n , and thus is infeasible to calculate in general. On the other hand, for mixed channels decomposed into stationary memoryless channels with a finite input alphabet, a single-letter characterization of the channel capacity has been given by Ahlswede [1]. This characterization is of importance because the channel capacity can be computed with the complexity independent of n . However, to the best of authors' knowledge, no single-letter characterizations of the ε -capacity have been known, or at least no rigorous proofs

of an expression have appeared in the literature. The regular decomposable channel which is decomposed into memoryless channels, introduced by Winkelbauer [15], is an example of channel classes for which a single-letter characterization of ε -capacity has been given.

This paper gives a single-letter characterization of the ε -capacity for mixed channels decomposed into stationary memoryless channels with a finite input alphabet and a general output alphabet. First, a single-letter characterization of the ε -capacity is given for mixed channels decomposed into at most countably many stationary memoryless channels¹. An alternative expression is also provided, and it is shown that the characterization reduces to the one for the channel capacity given by Ahlswede [1] when ε is zero. Then the theorem is extended to the case when input symbols are subject to a cost constraint. The coding theorems are proved by the *information spectrum method* (c.f. [5], [14]) combined with recently developed analytical methods for the finite blocklength regime (e.g., [6], [9], [11], [13]). In the proof of the coding theorems, the so-called *meta converse* bound [9], which is known as the best converse bound to date is particularized for mixed channels². With this bound, kinds of previously known converse bounds developed for general channels may also be particularized for the mixed channel setting.

II. PRELIMINARIES

A. General Channel and ε -Capacity

Consider a channel $W^n : \mathcal{X}^n \rightarrow \mathcal{Y}^n$ which stochastically maps an input sequence $X^n \in \mathcal{X}^n$ of length n into an output sequence $Y^n \in \mathcal{Y}^n$. Here, \mathcal{X} and \mathcal{Y} denote a finite input alphabet and an arbitrary output alphabet³, respectively. We

¹A single-letter expression of the capacity has also been given by Ahlswede [1] for the mixed channel averaged by an arbitrary probability measure, and the expression has been simplified by Han [5]. Other related studies which analyze the maximum rate for which the outage probability is admitted up to ε for a non-ergodic block fading channel has been given by [10] and [17].

²Although the meta converse bound also applies to mixed channels, it should be modified to finely analyze fundamental limits of codes.

³In the case where \mathcal{Y} is abstract in general, we understand that $W^n(\mathbf{y}|\mathbf{x})$ and $P_{Y^n}(\mathbf{y})$ denote the corresponding probability measures $W^n(d\mathbf{y}|\mathbf{x})$ and $P_{Y^n}(d\mathbf{y})$, respectively, and that $\log \frac{W^n(\mathbf{y}|\mathbf{x})}{P_{Y^n}(\mathbf{y})}$ denotes the Radon-Nikodym derivative $\log \frac{W^n(d\mathbf{y}|\mathbf{x})}{P_{Y^n}(d\mathbf{y})}$. As in [5], we keep the notation simple and use the summation \sum to denote the integral \int , too.

denote by $\mathcal{P}(\mathcal{X})$ the set of all probability mass functions on \mathcal{X} . A sequence $\mathbf{W} := \{W^n\}_{n=1}^\infty$ of channels W^n is referred to as a *general channel* [5].

Let \mathcal{C}_n be a code of length n and the number of codewords $|\mathcal{C}_n| = M_n$ with an encoding function $\phi : \{1, \dots, M_n\} \rightarrow \mathcal{X}^n$ and a decoding function $\psi : \mathcal{Y}^n \rightarrow \{1, \dots, M_n\}$.

Definition 1: The average probability of decoding error over W^n is defined as

$$P_e(\mathcal{C}_n) := \frac{1}{M_n} \sum_{i=1}^{M_n} \Pr[\psi(Y^n) \neq i | i \text{ sent}]. \quad (1)$$

The code \mathcal{C}_n is referred to as an $(n, M_n, P_e(\mathcal{C}_n))$ code. \square

Remark 1: The maximum error probability defined as

$$e(\mathcal{C}_n) := \max_{i \in \{1, \dots, M_n\}} \Pr[\psi(Y^n) \neq i | i \text{ sent}] \quad (2)$$

has also been considered in the literature. All the capacity results in this paper are also valid under the maximum error probability criterion. \square

Definition 2: A coding rate $R \geq 0$ is said to be *achievable* if there exists a sequence of $(n, M_n, P_e(\mathcal{C}_n))$ codes satisfying

$$\limsup_{n \rightarrow \infty} P_e(\mathcal{C}_n) \leq \varepsilon \quad \text{and} \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \log M_n \geq R. \quad (3)$$

The supremum of ε -achievable rates is called the ε -capacity and is denoted by $C(\varepsilon | \mathbf{W})$. \square

Remark 2: The ε -capacity $C(\varepsilon | \mathbf{W})$ is a right-continuous function in ε [14]. \square

Remark 3: An ε -achievable rate is often defined by replacing (3) with

$$P_e(\mathcal{C}_n) \leq \varepsilon \quad \text{and} \quad \frac{1}{n} \log M_n \geq R - \lambda \quad (4)$$

(e.g., [7], [14], [15], etc.). The ε -capacity in this case is not right-continuous in ε , and the provided characterizations of the ε -capacity are valid except at most countably many discontinuous points of ε -capacity (c.f. [14, Theorem 6]). \square

B. Mixed Memoryless Channel

Consider a set of at most countably many $\mathbf{W}_\ell := \{W_\ell^n\}_{n=1}^\infty$ ($\ell = 1, 2, \dots$), and the set of indices of \mathbf{W}_ℓ is denoted by Ω . The *mixed channel* decomposed into $\{\mathbf{W}_\ell\}_{\ell \in \Omega}$ is defined by

$$W^n(\mathbf{y} | \mathbf{x}) = \sum_{\ell \in \Omega} w_\ell W_\ell^n(\mathbf{y} | \mathbf{x}), \quad (\forall \mathbf{x} \in \mathcal{X}^n, \forall \mathbf{y} \in \mathcal{Y}^n), \quad (5)$$

a mixture of $\{W_\ell^n\}$ with the mixing ratio $\{w_\ell \geq 0\}_{\ell=1}^\infty$ satisfying $\sum_{\ell=1}^\infty w_\ell = \sum_{\ell \in \Omega} w_\ell = 1$. Hereafter, we assume that $w_\ell > 0$ for all $\ell \in \Omega$, for simplicity. Each \mathbf{W}_ℓ is called a *component channel* or simply components. Given an input probability distribution P_{X^n} , the output from W_ℓ^n induced by the input X^n is denoted by Y_ℓ^n . That is,

$$P_{X^n Y_\ell^n}(\mathbf{x}, \mathbf{y}) = P_{X^n}(\mathbf{x}) W_\ell^n(\mathbf{y} | \mathbf{x}) \quad (\forall \mathbf{x} \in \mathcal{X}^n, \forall \mathbf{y} \in \mathcal{Y}^n).$$

The mixed channel \mathbf{W} given by at most countably many stationary memoryless channels $\{\mathbf{W}_\ell\}_{\ell \in \Omega}$ satisfying $W_\ell^n(\mathbf{y} | \mathbf{x}) = \prod_{i=1}^n W_\ell(y_i | x_i)$ is called the **mixed memoryless channel**. Hereafter, we assume that the input alphabet \mathcal{X} is finite and the output alphabet \mathcal{Y} may be infinite as long as the mutual information $I_{P_X}(X; Y_\ell)$ calculated by P_X and W_ℓ

is continuous in P_X for all $\ell \in \Omega$. For example, if \mathcal{Y} is a complete separable metric space, then $I_{P_X}(X; Y_\ell)$ is concave and continuous in P_X [3, Lemma 3].

III. MAIN THEOREMS

A. General Mixed Memoryless Channels

The following theorem gives a single-letter characterization of the ε -capacity.

Theorem 1: Let \mathbf{W} be a mixed memoryless channel with $|\mathcal{X}| < \infty$. For any fixed $\varepsilon \in [0, 1)$, the ε -capacity is given by

$$C(\varepsilon | \mathbf{W}) = \sup_{P_X \in \mathcal{P}(\mathcal{X})} \sup \left\{ R \mid F_w(R | P_X) \leq \varepsilon \right\}, \quad (6)$$

where

$$F_w(R | P_X) := \sum_{\ell \in \Omega} w_\ell \mathbf{1} \{ I_{P_X}(X; Y_\ell) \leq R \}. \quad (7)$$

Here, $I_{P_X}(X; Y_\ell)$ denotes the mutual information calculated by P_X and W_ℓ , and $\mathbf{1}\{A\}$ denotes the indicator function which takes one if a proposition A is true and takes zero otherwise.

(Proof) A proof is given in Sect. V. \square

We define the function $A : \mathcal{P}(\mathcal{X}) \times [0, 1] \rightarrow \mathbb{R}$ as

$$A(P_X, \delta) := \sup \{ R \mid F_w(R | P_X) \leq \delta \}, \quad (8)$$

where \mathbb{R} denotes the set of real numbers. The ε -capacity given by Theorem 1 is expressed as

$$C(\varepsilon | \mathbf{W}) = \sup_{P_X \in \mathcal{P}(\mathcal{X})} A(P_X, \varepsilon). \quad (9)$$

Let \mathcal{D} be a compact set in $\mathcal{P}(\mathcal{X})$. Some properties of the function $A(P_X, \delta)$ and $\tilde{R}(\delta | \mathcal{D}) := \sup_{P_X \in \mathcal{D}} A(P_X, \delta)$ are shown by the following lemma.

Lemma 1: For the functions $A(P_X, \delta)$ and $\tilde{R}(\delta | \mathcal{D})$, the following hold:

- (a) $A(P_X, \delta)$ is continuous in P_X for fixed δ .
- (b) $A(P_X, \delta)$ is non-decreasing in δ for fixed P_X .
- (c) $A(P_X, \delta)$ is right-continuous in δ . That is, $\lim_{\delta \downarrow \delta_0} A(P_X, \delta) = A(P_X, \delta_0)$.
- (d) $\tilde{R}(\delta | \mathcal{D})$ is right-continuous in δ .

(Proof) Properties (b) and (c) are easily verified by the definition of $A(P_X, \delta)$. Proofs of Properties (a) and (d) are given in Appendix A-A and Appendix A-B, respectively. \square

The function $F_w(R | P_X)$ which appears in the definition of $A(P_X, \delta)$ is not continuous in P_X obviously. It is of interest to see that the function $A(P_X, \delta)$ has Property (a) nevertheless. By Property (a), there exists at least one $P_X \in \mathcal{P}(\mathcal{X})$ that gives $\tilde{R}(\delta | \mathcal{D})$. That is, $\tilde{R}(\delta | \mathcal{D}) = \max_{P_X \in \mathcal{D}} A(P_X, \delta)$. The first supremum on the right-hand side (r.h.s.) of (6) is actually maximum.

By Properties (b) and (d), the ε -capacity given in Theorem 1 can also be expressed as

$$C(\varepsilon | \mathbf{W}) = \inf_{\delta > 0} \sup_{P_X \in \mathcal{P}(\mathcal{X})} \sup \{ R \mid F_w(R | P_X) \leq \varepsilon + \delta \} \quad (10)$$

To prove Theorem 1, it is sufficient to show that (10) holds, and this fact is used in Sect. V.

B. Special Case: Well-Ordered Mixed Memoryless Channels

It is shown that the single-letter characterization in Theorem 1 reduces a previously known expression for a restricted class of mixed channels. As an example, the following class of mixed memoryless channels is introduced.

Let C_ℓ denote the channel capacity of the ℓ -th component channel W_ℓ and Π_ℓ be the set of input probability distributions that achieve C_ℓ . Without loss of generality, we assume that the component channels are indexed to satisfy $C_\ell \leq C_{\ell+1}$, where components W_i and W_j ($i \neq j$) such that $C_i = C_j$ are arbitrarily indexed if $|\Omega| < \infty$.

Definition 3 (Well-Ordered Mixed Memoryless Channel): For each $\ell \in \Omega$, if there exists some $P_X \in \Pi_\ell$ such that

$$C_\ell \leq I_{P_X}(X; Y_j) \text{ for all } j : C_\ell \leq C_j, \quad (11)$$

then the mixed channel \mathbf{W} is said to be **well-ordered**. \square

For example, let us consider a well-ordered mixed memoryless channel of two components $\mathbf{W}_1 = \{W_1^n\}_{n=1}^\infty$, $\mathbf{W}_2 = \{W_2^n\}_{n=1}^\infty$. By the condition (11), it should hold $C_1 \leq C_2$ and

$$C_1 \leq I(X; Y_2) \quad (\exists P_X \in \Pi_1). \quad (12)$$

When $C_1 = C_2$, (11) requires $\Pi_1 \cap \Pi_2 \neq \emptyset$. If the component channels $\{W_\ell | \ell \in \Omega\}$ are all output-symmetric (e.g., the mixed BSCs [7]), then the condition (11) trivially holds.

It is readily shown that every well-ordered mixed memoryless channel is an instance of *regular decomposable* channels introduced by Winkelbauer [15]. The ε -capacity of a regular decomposable channel has been given by [15]. For well-ordered mixed memoryless channels, the following corollary follows from Theorem 1.

Corollary 1 (Winkelbauer [15]): Let \mathbf{W} be a well-ordered mixed memoryless channels such that $|\mathcal{X}| < \infty$, and define

$$\tilde{F}_w(R) := \sum_{\ell \in \Omega} w_\ell \mathbf{1}\{C_\ell \leq R\}. \quad (13)$$

For any $\varepsilon \in [0, 1]$, the ε -capacity is given by

$$C(\varepsilon | \mathbf{W}) = \sup \left\{ R \mid \tilde{F}_w(R) \leq \varepsilon \right\}, \quad (14)$$

\square

Corollary 1 slightly extends the coding theorem by Winkelbauer [15] for the well-ordered mixed memoryless channel to the case of non-discrete \mathcal{Y} .

Consider the case $|\Omega| < \infty$. By (14), the ε -capacity of the mixed channel satisfying (11) is given by $C(\varepsilon | \mathbf{W}) = C_{k^*}$, where k^* is the component index satisfying

$$\sum_{\ell \in \Omega} w_\ell \mathbf{1}\{C_\ell < C_{k^*}\} \leq \varepsilon < \tilde{F}_w(C_{k^*}). \quad (15)$$

For example, the ε -capacity for the well-ordered mixed channel \mathbf{W} with $|\Omega| = 3$ is given by

$$C(\varepsilon | \mathbf{W}) = \begin{cases} C_1, & \text{if } \varepsilon \in [0, w_1) \\ C_2, & \text{if } \varepsilon \in [w_1, w_1 + w_2) \\ C_3, & \text{otherwise} \end{cases}. \quad (16)$$

It is of interest to see that the expression of the ε -capacity in Corollary 1 is similar to the one for the channel with states [12]. Specifically, Example 1 in [12] deals with the mixed channel decomposable into finitely many (not necessarily

well-ordered) stationary memoryless components, and both the encoder and the decoder can access the channel state information, which corresponding to the index of component channels in this paper. In this case, the expression of the ε -capacity coincides with the one given in (14). This fact implies that the optimum rate without the channel state information is the same as the one with the channel state information if the mixed channel is well-ordered.

C. Alternative Expression of ε -Capacity

We give an alternative expression of the ε -capacity of the mixed memoryless channel given by Theorem 1. We first show the following lemma.

Lemma 2: Let \mathbf{W} be a mixed memoryless channel with $|\mathcal{X}| < \infty$. Then, we have

$$\sup \{ R \mid F_w(R | P_X) \leq \varepsilon \} = \sup_{\{S \subseteq \Omega \mid w(S) \geq 1-\varepsilon\}} \inf_{\ell \in S} I_{P_X}(X; Y_\ell) \quad (17)$$

for all $P_X \in \mathcal{P}(\mathcal{X})$, where $w(S)$ denotes $\sum_{\ell \in S} w_\ell$.

(Proof) See Appendix B. \square

Combining (6) with Lemma 2 provides an alternative expression of the ε -capacity as

$$C(\varepsilon | \mathbf{W}) = \sup_{P_X} \sup_{\{S \subseteq \Omega \mid w(S) \geq 1-\varepsilon\}} \inf_{\ell \in S} I_{P_X}(X; Y_\ell) \quad (18)$$

in the case of at most countably many component channels. When $\varepsilon = 0$, the r.h.s. of (18) becomes $\sup_{P_X} \inf_{\ell \in \Omega} I_{P_X}(X; Y_\ell)$, which coincides with the capacity expression given by Ahlswede [1].

On the r.h.s. of (18), $\inf_{\ell \in S} I_{P_X}(X; Y_\ell)$ with $w(S) \geq 1-\varepsilon$ is the infimum of concave functions of P_X . When $\varepsilon = 0$, $\inf_{\ell \in S} I_{P_X}(X; Y_\ell) = \inf_{\ell \in \Omega} I_{P_X}(X; Y_\ell)$ is concave in P_X . When $\varepsilon > 0$, however, this function is not necessarily concave since the domain S with $w(S) \geq 1-\varepsilon$ depends on P_X .

Similar to (18), the ε -capacity of a well-ordered mixed memoryless channel can also be expressed as

$$C(\varepsilon | \mathbf{W}) = \sup_{\{S \subseteq \Omega \mid w(S) \geq 1-\varepsilon\}} \inf_{\ell \in S} C_\ell. \quad (19)$$

D. ε -Capacity under Cost Constraint

We now turn to considering the coding for which an input symbol X is constrained by a cost function $c : \mathcal{X} \rightarrow \mathbb{R}$. This problem includes the power constraint over the channel with a continuous alphabet such as the additive white Gaussian noise (AWGN) channel as an instance.

If every codeword $\phi(i)$ ($\forall i \in \{1, \dots, M_n\}$) of a code \mathcal{C}_n is restricted to be in the set

$$\mathcal{X}_{c,\Gamma} := \left\{ \mathbf{x} \in \mathcal{X}^n \mid \sum_{i=1}^n c(x_i) \leq n\Gamma \right\}, \quad (20)$$

this condition is referred to as the *cost constraint* Γ . A code \mathcal{C}_n attains an error probability $\varepsilon \in [0, 1]$ under the cost constraint Γ is called an $(n, M_n, \varepsilon, \Gamma)$ code.

Definition 4: If (3) holds under the cost constraint Γ , then the rate R is said to be (ε, Γ) -achievable. The supremum of (ε, Γ) -achievable rates for \mathbf{W} is referred to as the (ε, Γ) -capacity and is denoted by $C(\varepsilon, \Gamma | \mathbf{W})$. \square

The following theorem characterizes the optimum coding rate under a cost constraint for the mixed memoryless channel.

Theorem 2: Let \mathbf{W} be a mixed memoryless channel with $|\mathcal{X}| < \infty$. The (ε, Γ) -capacity for a given $\Gamma \in \mathbb{R}$ and $\varepsilon \in [0, 1]$ is given by

$$C(\varepsilon, \Gamma | \mathbf{W}) = \sup_{P_X: E_{P_X} c(X) \leq \Gamma} \sup \left\{ R \mid F_w(R | P_X) \leq \varepsilon \right\}. \quad (21)$$

(Proof) Converse Part is exactly the same line as the one for Theorem 1. To prove Direct Part, we use an ensemble of constant composition codes whose type P_0 satisfies the constraint Γ and M_n codeword are chosen by the uniform distribution on the set of sequences with type P_0 . We can apply an information spectrum approach by Hayashi [6, Sect. X-B] to the proof of Direct Part of Theorem 1, showing that any rate R less than the r.h.s. of (21) is (ε, Γ) -achievable. \square

The set of $P_X \in \mathcal{P}(\mathcal{X})$ such that $E_{P_X} c(X) \leq \Gamma$ is closed convex, and hence is compact. Then from Property (a) in Lemma 1, the first supremum in (21) is maximum, and from Property (d), the r.h.s in (21) is right-continuous in ε . When $\varepsilon = 0$, (21) reduces to the capacity under a cost constraint Γ :

$$C(0, \Gamma | \mathbf{W}) = \sup_{P_X: E_{P_X} c(X) \leq \Gamma} \inf_{\ell \in \Omega} I_{P_X}(X; Y_\ell), \quad (22)$$

which has been shown by Han [5].

The function $C(\varepsilon, \Gamma | \mathbf{W})$ is referred to as the *capacity-cost function*, which is analogous to the rate-distortion function for lossy source coding (c.f. [5]). The capacity-cost function is also referred to as the *capacity-expense function*, and some of its properties for discrete memoryless channels (DMCs) have been shown in [2]. By definition, the capacity-cost function is monotonic nondecreasing in Γ . We show some properties of the capacity-cost function.

Theorem 3: The capacity-cost function has the following properties:

- (i) concave in Γ for $\Gamma > 0$;
- (ii) strictly increasing in Γ for $0 \leq \Gamma < \Gamma^*$, where Γ^* is the minimum cost for which the capacity cost-function coincides with the ε -capacity;
- (iii) if $\Gamma < \Gamma^*$, then $C(\varepsilon, \Gamma | \mathbf{W})$ is achieved by some $P_X \in \mathcal{P}(\mathcal{X})$ such that $E_{P_X} c(X) = \Gamma$. \square

These properties, which can be shown in an analogous way to the proofs in [2, Appendix], are handed down from the capacity-cost function for DMCs. However, unlike the DMC case, the set of optimum input distributions that achieve the ε -capacity under a cost constraint is not necessarily convex.

IV. ONE-SHOT ERROR BOUNDS FOR MIXED CHANNEL

The proof of Theorem 1 provided in Sect. V uses so-called “one-shot” error bounds which hold for the mixed channel decomposed into (not necessarily stationary or ergodic) general component channels.

First we show converse (lower) error bounds. Following [9, Sect. III-D], we introduce simple hypothesis testing: Given an observation $Z \in \mathcal{Z}$ according to either of two probability

measures P, Q on \mathcal{Z} , consider a hypothesis test

$$H_0 : Z \sim P \text{ vs. } H_1 : Z \sim Q \quad (23)$$

to judge the true probability measure. When observing Z , a test $\xi : \mathcal{Z} \rightarrow \{0, 1\}$ judges P to be true with probability $\xi(Z)$ and Q to be true with probability $1 - \xi(Z)$. The error event when the true measure is P is called *the error of the first kind* and the one when the true measure is Q is called *the error of the second kind*. For a fixed $\alpha \in [0, 1]$, the optimum test that minimizes the error probability of the second kind among those whose error probability of the first kind satisfies $\sum_{z \in \mathcal{Z}} P(z)(1 - \xi(z)) \leq \alpha$ is denoted by ξ^* , and its error probability of the second kind is denoted by

$$\beta_\alpha(P, Q) := \min_{\substack{\xi: \mathcal{Z} \rightarrow \{0, 1\}: \\ \sum_z P(z)(1 - \xi(z)) \leq \alpha}} \sum_{z \in \mathcal{Z}} Q(z)\xi(z). \quad (24)$$

Likewise, let $\alpha_\beta(P, Q)$ denote the minimum error probability of the second kind among tests whose error probability of the first kind is less than or equal to β .

The following lemma particularizes a meta converse bound by Polyanskiy, Poor, and Verdú [9] for the mixed channels.

Lemma 3 (Meta Converse for Mixed Channel): Let $\{Q_{Y_\ell^n}\}_{\ell \in \Omega}$ be a set of arbitrary probability measures. Then every (n, M_n, ε_n) code \mathcal{C}_n with a (possibly probabilistic) decoding function $\xi : \mathcal{Y}^n \rightarrow \{1, 2, \dots, M_n\}$ satisfies

$$\varepsilon_n \geq \sum_{\ell \in \Omega} w_\ell \alpha_{\frac{1}{M_n}}(P_{X^n} W_\ell^n, P_{X^n} Q_{Y_\ell^n}) \quad (25)$$

and

$$\frac{1}{M_n} \geq \sum_{\ell \in \Omega} w_\ell \beta_{\varepsilon_n^{(\ell)}}(P_{X^n} W_\ell^n, P_{X^n} Q_{Y_\ell^n}). \quad (26)$$

Here, P_{X^n} is the uniform distribution on \mathcal{C}_n , and $\varepsilon_n^{(\ell)}$ denotes the average probability of decoding error over W_ℓ^n given by

$$\varepsilon_n^{(\ell)} := 1 - \frac{1}{M_n} \sum_{i=1}^{M_n} \sum_{\mathbf{y} \in \mathcal{Y}^n} W_\ell^n(\mathbf{y} | \phi(i)) \xi(i | \mathbf{y}) \quad (\forall \ell \in \Omega), \quad (27)$$

where $\phi(i)$ denotes the codeword assigned to message i , and $\xi(i | \mathbf{y})$ denotes the probability of i being estimated given \mathbf{y} . (Proof) The first inequality is due to [13], and the second one is due to [9]. A proof is given in Appendix C. \square

The following lemma is established by modifying a lemma shown by Tomamichel and Tan [11] for mixed channels.

Lemma 4: Given a family of pairs of probability measures $\{(P_\ell, Q_\ell)\}_{\ell \in \Omega}$ on \mathcal{Z} , consider a hypothesis test

$$H_0 : Z_\ell \sim P_\ell \text{ vs. } H_1 : Z_\ell \sim Q_\ell \quad (28)$$

for each $\ell \in \Omega$. For any given $\varepsilon \in [0, 1]$, letting $\{\varepsilon_\ell \in [0, 1]\}_{\ell \in \Omega}$ be a sequence such that $\sum_{\ell \in \Omega} w_\ell \varepsilon_\ell = \varepsilon$, we have

$$-\log \sum_{\ell \in \Omega} w_\ell \beta_{\varepsilon_\ell}(P_\ell, Q_\ell) \leq D_s^{\varepsilon+\delta}(\{P_\ell\} || \{Q_\ell\}) - \log \delta \quad (29)$$

with an arbitrary constant $\delta \in (0, 1]$, where $D_s^\varepsilon(\{P_\ell\} || \{Q_\ell\})$ denotes the value

$$\sup \left\{ R \mid \sum_{\ell \in \Omega} w_\ell P_\ell \left\{ \log \frac{P_\ell(Z_\ell)}{Q_\ell(Z_\ell)} \leq R \right\} \leq \varepsilon \right\}. \quad (30)$$

(Proof) A proof is given in Appendix D. \square

We set $P_\ell := P_{X^n} \times W_\ell^n, Q_\ell := P_{X^n} \times Q_{Y_\ell^n}, \varepsilon := \varepsilon_n$, and $\varepsilon_\ell := \varepsilon_n^{(\ell)}$ in Lemma 4. Since $\varepsilon_n^{(\ell)}$ given in (27) satisfies $\sum_{\ell \in \Omega} w_\ell \varepsilon_n^{(\ell)} = \varepsilon_n$, (29) holds. Then from (26), every (n, M_n, ε_n) code \mathcal{C}_n satisfies

$$\log M_n \leq D_s^{\varepsilon_n + \delta}(\{P_{X^n} W_\ell^n\} || \{P_{X^n} Q_{Y_\ell^n}\}) - \log \delta \quad (31)$$

with an arbitrary constant $\delta \in (0, 1]$.

Remark 4: It is easily verified that Lemmas 3 and 4 can be extended to the mixed channel with a general mixture (c.f. [5, Sect. 3.3]). In this case, the summand should be replaced with integral. \square

We next consider upper (achievability) error bounds. The following lemma particularizes the Feinstein upper bound [4] for the mixed channels.

Lemma 5: For any given $P_{X^n} \in \mathcal{P}(\mathcal{X}^n)$, there exists an (n, M_n, ε_n) code satisfying

$$\varepsilon_n \leq \sum_{\ell \in \Omega} w_\ell \Pr \left\{ \frac{1}{n} \log \frac{W_\ell^n(Y_\ell^n | X^n)}{P_{Y_\ell^n}(Y_\ell^n)} \leq \frac{1}{n} \log M_n + \gamma + \frac{1}{n} \log \frac{1}{w_\ell} \right\} + e^{-n\gamma}, \quad (32)$$

where $\gamma > 0$ is an arbitrary constant and P_{Y^n} denotes the marginal measure $P_{Y^n}(\mathbf{y}) = \sum_{\mathbf{x} \in \mathcal{X}^n} P_{X^n}(\mathbf{x}) W_\ell^n(\mathbf{y} | \mathbf{x})$. \square

Equation (32) can be derived by the result shown by Han [5, Lemma 1.4.1]. Although the original bound by Han uses a sequence $\{\gamma_n \geq 0 | \lim_{n \rightarrow \infty} \gamma_n = 0\}$ instead of a constant γ , an examination verifies that (32) holds for any constant $\gamma > 0$.

V. PROOF OF THEOREM 1

A. Converse Part

For a given $\mathbf{x} \in \mathcal{X}^n$, we denote $W_{\ell|\mathbf{x}}^n := W_\ell^n(\cdot | \mathbf{x})$ for simplicity. For a given $P_X \in \mathcal{P}(\mathcal{X})$, we define

$$(P_X W_\ell)^{\times n}(\mathbf{y}) := \prod_{i=1}^n \sum_{x \in \mathcal{X}} P_X(x) W_\ell(y_i | x). \quad (33)$$

Converse Part of Theorem 1 is stated as follows:

Theorem 4 (Converse Theorem): For a mixed channel \mathbf{W} , any ε -achievable rate R for $\varepsilon \in [0, 1)$ satisfies

$$R \leq \inf_{\delta > 0} \sup_{P_X \in \mathcal{P}(\mathcal{X})} \sup \left\{ R \mid F_w(R | P_X) \leq \varepsilon + \delta \right\}. \quad (34)$$

\square

Before stating the proof of Converse Part, we give some preliminaries. By the Chebyshev inequality, the following lemma holds:

Lemma 6: For any fixed $\mathbf{x} \in \mathcal{X}^n$, we denote its type (empirical distribution) by P_n . Let $\gamma > 0$ be an arbitrary constant and define

$$\mathcal{B}_{\ell|\mathbf{x}}^{(n)}(\gamma) := \left\{ \mathbf{y} \mid \left| \frac{1}{n} \log \frac{W_\ell^n(\mathbf{y} | \mathbf{x})}{(P_n W_\ell)^{\times n}(\mathbf{y})} - I_{P_n}(X; Y_\ell) \right| \leq \gamma \right\} \quad (35)$$

for all $\ell \in \Omega$. Then we have

$$W_{\ell|\mathbf{x}}^n \left\{ Y_\ell^n \in \mathcal{B}_{\ell|\mathbf{x}}^{(n)}(\gamma) \right\} \geq 1 - \frac{A(\gamma)}{n} \quad (36)$$

with a constant $A(\gamma) \geq 0$ independent of n, P_n and ℓ . \square

The conditional variance of information density $\log \frac{W_\ell(Y_\ell | X)}{(P_X W_\ell)(Y_\ell)}$ given P_X ,

$$V(P_X, W_\ell) := \mathbb{E}_{P_X} \left[\mathbb{V}_{W_\ell} \left[\log \frac{W_\ell(Y_\ell | X)}{(P_X W_\ell)(Y_\ell)} \mid X \right] \right], \quad (37)$$

is upper bounded by $\mathbb{V}_{P_X W_\ell} \left[\log \frac{W_\ell(Y_\ell | X)}{(P_X W_\ell)(Y_\ell)} \right]$, which can be verified as follows (see also [9, Lemma 62]): defining

$$U_1 := \mathbb{E} \left[\mathbb{E} \left[\log \frac{W_\ell(Y_\ell | X)}{(P_X W_\ell)(Y_\ell)} \mid X \right]^2 \right],$$

$$U_2 := \mathbb{E} \left[\log \frac{W_\ell(Y_\ell | X)}{(P_X W_\ell)(Y_\ell)} \right]^2, \quad (38)$$

then $\mathbb{E} \left[\log \frac{W_\ell(Y_\ell | X)}{(P_X W_\ell)(Y_\ell)} \mid X \right]^2$ is a convex function of P_X since $f(z) := z^2$ is convex and nondecreasing for $z \geq 0$, and $g(x) := \mathbb{E} \left[\log \frac{W_\ell(Y_\ell | x)}{(P_X W_\ell)(Y_\ell)} \right]$ is convex. Therefore, we obtain $U_1 \geq U_2$, which leads to the claim. The variance $\mathbb{V}_{P_X W_\ell} \left[\log \frac{W_\ell(Y_\ell | X)}{(P_X W_\ell)(Y_\ell)} \right]$ is further bounded uniformly by $\frac{8|\mathcal{X}|}{\varepsilon^2}$ [5, Remark 3.1.1], the constant $A(\gamma)$ in (36) can be chosen independently of $\ell \in \Omega$ and $P_n \in \mathcal{T}_n$.

We are now in a position to prove Theorem 4. Let R be ε -achievable. Then, from (3), there exists a sequence of (n, M_n, ε_n) codes \mathcal{C}_n with some $\{\delta_n \geq 0 | \delta_1 \geq \delta_2 \geq \dots \geq 0, \lim_{n \rightarrow \infty} \delta_n = 0\}$ satisfying

$$\frac{1}{n} \log M_n \geq R - \gamma \text{ and } \varepsilon_n \leq \varepsilon + \delta_n \text{ } (\exists n_1 > 0; \forall n \geq n_1) \quad (39)$$

for an arbitrarily fixed constant $\gamma > 0$. Borrowing an idea given by Hayashi [6, Sect. X-A], we set $\delta = \frac{1}{n}$ and

$$Q_{Y_\ell^n}(\mathbf{y}) = \frac{1}{|\mathcal{T}_n|} \sum_{P_n \in \mathcal{T}_n} (P_n W_\ell)^{\times n}(\mathbf{y}) \quad (\forall \mathbf{y} \in \mathcal{Y}^n) \quad (40)$$

in (31), where \mathcal{T}_n denotes the set of types on \mathcal{X}^n . We define

$$R_n^* := \frac{1}{n} D_s^{\varepsilon_n + \frac{1}{n}}(\{P_{X^n} W_\ell^n\} || \{P_{X^n} Q_{Y_\ell^n}\}) + \frac{1}{n} \log n. \quad (41)$$

Since the first term on the r.h.s. is expressed as

$$\sup \left\{ R \mid \sum_{\ell \in \Omega} w_\ell \Pr \left\{ \frac{1}{n} \log \frac{W_\ell^n(Y_\ell^n | X^n)}{Q_{Y_\ell^n}(Y_\ell^n)} \leq R \right\} \leq \varepsilon_n + \frac{1}{n} \right\},$$

it can be verified that there exists an $\mathbf{x}_0 \in \mathcal{C}_n$ such that

$$\sum_{\ell \in \Omega} w_\ell W_{\ell|\mathbf{x}_0}^n \left\{ \frac{1}{n} \log \frac{W_\ell^n(Y_\ell^n | \mathbf{x}_0)}{Q_{Y_\ell^n}(Y_\ell^n)} \leq R_n^* - \frac{1}{n} \log n - \gamma \right\} \leq \varepsilon_n + \frac{1}{n} \quad (42)$$

as follows: By definition in (41), we can re-express

$$\begin{aligned}
& R_n^* - \frac{1}{n} \log n \\
&= \sup \left\{ R \left| \sum_{\ell \in \Omega} w_\ell \Pr \left\{ \frac{1}{n} \log \frac{W_\ell^n(Y_\ell^n | X^n)}{Q_{Y_\ell^n}(Y_\ell^n)} \leq R \right\} \leq \varepsilon_n + \frac{1}{n} \right\} \right. \\
&= \sup \left\{ R \left| \sum_{\mathbf{x} \in \mathcal{C}_n} \frac{1}{M_n} \sum_{\ell \in \Omega} w_\ell W_{\ell|\mathbf{x}}^n \left\{ \frac{1}{n} \log \frac{W_\ell^n(Y_\ell^n | \mathbf{x})}{Q_{Y_\ell^n}(Y_\ell^n)} \leq R \right\} \right. \right. \\
&\quad \left. \left. \leq \varepsilon_n + \frac{1}{n} \right\} \right\}. \quad (43)
\end{aligned}$$

Suppose that (42) does not hold for any $\mathbf{x} \in \mathcal{C}_n$. Then we have

$$\begin{aligned}
& \frac{1}{M_n} \sum_{\mathbf{x} \in \mathcal{C}_n} \sum_{\ell \in \Omega} w_\ell W_{\ell|\mathbf{x}_0}^n \left\{ \frac{1}{n} \log \frac{W_\ell^n(Y_\ell^n | \mathbf{x}_0)}{Q_{Y_\ell^n}(Y_\ell^n)} \right. \\
&\quad \left. \leq R_n^* - \frac{1}{n} \log n - \gamma \right\} > \varepsilon_n + \frac{1}{n}, \quad (44)
\end{aligned}$$

and this implies that $R_n^* - \frac{1}{n} \log n - \gamma$ is strictly greater than the r.h.s. of (43). Since this contradicts (43), it is concluded that there exists at least one $\mathbf{x}_0 \in \mathcal{C}_n$ satisfying (42).

Denoting by P_0^n the type of \mathbf{x}_0 , we have a chain of inequalities

$$\begin{aligned}
& W_{\ell|\mathbf{x}_0}^n \left\{ \frac{1}{n} \log \frac{W_\ell^n(Y_\ell^n | \mathbf{x}_0)}{Q_{Y_\ell^n}(Y_\ell^n)} \leq R_n^* - \frac{1}{n} \log n - \gamma \right\} \\
&\geq W_{\ell|\mathbf{x}_0}^n \left\{ \frac{1}{n} \log \frac{W_\ell^n(Y_\ell^n | \mathbf{x}_0)}{(P_0^n W_\ell)^{\times n}(Y_\ell^n)} \leq R_n^* - \frac{1}{n} \log n |\mathcal{T}_n| - \gamma \right\} \\
&\geq W_{\ell|\mathbf{x}_0}^n \left\{ \frac{1}{n} \log \frac{W_\ell^n(Y_\ell^n | \mathbf{x}_0)}{(P_0^n W_\ell)^{\times n}(Y_\ell^n)} \leq R_n^* - \frac{1}{n} \log n |\mathcal{T}_n| - \gamma, \right. \\
&\quad \left. Y_\ell^n \in \mathcal{B}_{\ell|\mathbf{x}_0}^{(n)}(\gamma) \right\} \\
&\geq \mathbf{1} \left\{ I_{P_0^n}(X; Y_\ell) \leq R_n^* - \frac{1}{n} \log n |\mathcal{T}_n| - 2\gamma \right\} - \frac{A(\gamma)}{n}, \quad (45)
\end{aligned}$$

where $\mathcal{B}_{\ell|\mathbf{x}_0}^{(n)}(\gamma)$ is defined in (35) and $A(\gamma) \geq 0$ is a constant independent of n , P_0^n , and ℓ . We use the relation in (40) for the first inequality. The inequality in (45) can be verified since (i) for $\ell \in \Omega$ such that $I_{P_0^n}(X; Y_\ell) \leq R_n^* - \frac{1}{n} \log n |\mathcal{T}_n| - 2\gamma$, we have

$$\begin{aligned}
& W_{\ell|\mathbf{x}_0}^n \left\{ \frac{1}{n} \log \frac{W_\ell^n(Y_\ell^n | \mathbf{x}_0)}{(P_0^n W_\ell)^{\times n}(Y_\ell^n)} \leq R_n^* - \frac{1}{n} \log n |\mathcal{T}_n| - \gamma, \right. \\
&\quad \left. Y_\ell^n \in \mathcal{B}_{\ell|\mathbf{x}_0}^{(n)}(\gamma) \right\} \\
&= W_{\ell|\mathbf{x}_0}^n \left\{ Y_\ell^n \in \mathcal{B}_{\ell|\mathbf{x}_0}^{(n)}(\gamma) \right\} \geq 1 - \frac{A(\gamma)}{n} \quad (46)
\end{aligned}$$

by Lemma 6 and (ii) for $\ell \in \Omega$ such that $I_{P_0^n}(X; Y_\ell) > R_n^* - \frac{1}{n} \log n |\mathcal{T}_n| - 2\gamma$, a trivial lower bound

$$\begin{aligned}
& W_{\ell|\mathbf{x}_0}^n \left\{ \frac{1}{n} \log \frac{W_\ell^n(Y_\ell^n | \mathbf{x}_0)}{(P_0^n W_\ell)^{\times n}(Y_\ell^n)} \leq R_n^* - \frac{1}{n} \log n |\mathcal{T}_n| - \gamma, \right. \\
&\quad \left. Y_\ell^n \in \mathcal{B}_{\ell|\mathbf{x}_0}^{(n)}(\gamma) \right\} \geq -\frac{A(\gamma)}{n} \quad (47)
\end{aligned}$$

holds. Note that the r.h.s. of (45) depends on $P_0^n \in \mathcal{T}_n$ but not on individual codewords. Since $A(\gamma) \geq 0$ is a constant

independent of ℓ and P_0^n , we obtain

$$\begin{aligned}
& \sum_{\ell} w_\ell \mathbf{1} \left\{ I_{P_0^n}(X; Y_\ell) \leq R_n^* - \frac{1}{n} \log n |\mathcal{T}_n| - 2\gamma \right\} \\
&\leq \varepsilon_n + \frac{1}{n} + \frac{A(\gamma)}{n} \quad (48)
\end{aligned}$$

from (42) and (45).

Combining (31), (39), and (41) gives

$$R - \gamma \leq R_n^* \quad (\forall n \geq n_1). \quad (49)$$

Then (48) implies that there exists a sequence of types $\{P_n \in \mathcal{T}_n\}_{n=n_1}^\infty$ such that

$$\begin{aligned}
& \sum_{\ell \in \Omega} w_\ell \mathbf{1} \left\{ I_{P_n}(X; Y_\ell) \leq R - 3\gamma - \frac{1}{n} \log n |\mathcal{T}_n| \right\} \\
&\leq \varepsilon + \delta_n + \frac{1}{n} + \frac{A(\gamma)}{n} \quad (50)
\end{aligned}$$

holds for all $n \geq n_1$, where the relation $\varepsilon_n \leq \varepsilon + \delta_n$ ($\forall n \geq n_1$) in (39) is used. Setting $\rho_n := \delta_n + \frac{1}{n} + \frac{A(\gamma)}{n}$, we obtain

$$\sum_{\ell \in \Omega} w_\ell \mathbf{1} \left\{ I_{P_n}(X; Y_\ell) \leq R - 3\gamma - \frac{1}{n} \log n |\mathcal{T}_n| \right\} \leq \varepsilon + \rho_n \quad (51)$$

for $n \geq n_1$.

It can be verified from (51) and the definition of $\tilde{R}(\cdot) := \tilde{R}(\cdot | \mathcal{P}(\mathcal{X}))$ that

$$R - 3\gamma - \frac{1}{n} \log n |\mathcal{T}_n| \leq \tilde{R}(\varepsilon + \rho_n) \quad (52)$$

holds for $n \geq n_1$. It is well-known that $|\mathcal{T}_n| \leq (n+1)^{|\mathcal{X}|}$ holds by the method of types, and taking the limes superior with respect to n on both sides of (52) yields

$$R - 3\gamma \leq \lim_{n \rightarrow \infty} \tilde{R}(\varepsilon + \rho_n) = \inf_{\delta > 0} \tilde{R}(\varepsilon + \delta). \quad (53)$$

The equality in (53) is due to Property (d) in Lemma 1. Since $\gamma > 0$ is an arbitrary constant, (53) implies $R \leq \inf_{\delta > 0} \tilde{R}(\varepsilon + \delta)$, i.e., (34).

B. Direct Part

Direct Part of Theorem 1 is stated as follows:

Theorem 5 (Direct Theorem): Let \mathbf{W} be a mixed memory-less channel such that $|\mathcal{X}| < \infty$. For a fixed $\varepsilon \in [0, 1)$, any rate R satisfying

$$R < \inf_{\delta > 0} \sup_{P_X \in \mathcal{P}(\mathcal{X})} \sup \left\{ R \left| F_w(R | P_X) \leq \varepsilon + \delta \right. \right\} \quad (54)$$

is ε -achievable. \square

The following lemma is used to prove Direct Part.

Lemma 7: Let P_{X^n} be a product distribution of a given $P_X \in \mathcal{P}(\mathcal{X})$. Then we have

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} \Pr \left\{ \frac{1}{n} \log \frac{W_\ell^n(Y_\ell^n | X^n)}{(P_X W_\ell)^{\times n}(Y_\ell^n)} \leq R + \rho_{\ell,n} \right\} \\
&\leq \mathbf{1} \{ I_{P_X}(X; Y_\ell) \leq R + \gamma \} \quad (\forall \ell \in \Omega), \quad (55)
\end{aligned}$$

where $\{\rho_{\ell,n} \geq 0\}$ denotes an arbitrary sequence such that $\lim_{n \rightarrow \infty} \rho_{\ell,n} = 0$, and $\gamma > 0$ denotes an arbitrary constant.

(Proof) See Appendix E. \square

We now prove Direct Part. Setting

$$R_0 := \inf_{\delta > 0} \sup_{P_X \in \mathcal{P}(\mathcal{X})} \sup \left\{ R \mid F_w(R|P_X) \leq \varepsilon + \delta \right\}, \quad (56)$$

we shall show that $R := R_0 - 4\gamma$ is ε -achievable for any $\gamma > 0$.

Fix $\gamma > 0$ arbitrarily. By (56), we have

$$R_0 \leq \sup_{P_X \in \mathcal{P}(\mathcal{X})} \sup \left\{ R \mid F_w(R|P_X) \leq \varepsilon + \delta \right\} \quad (57)$$

for all $\delta > 0$. For an arbitrarily fixed $\delta > 0$, there exists a $P_X^{(\delta)} \in \mathcal{P}(\mathcal{X})$ such that

$$\begin{aligned} & \sup_{P_X \in \mathcal{P}(\mathcal{X})} \sup \left\{ R \mid F_w(R|P_X) \leq \varepsilon + \delta \right\} \\ & \leq \sup \left\{ R \mid F_w(R|P_X^{(\delta)}) \leq \varepsilon + \delta \right\} + \gamma. \end{aligned} \quad (58)$$

It follows from (57) and (58) that

$$\sup \left\{ R \mid F_w(R|P_X^{(\delta)}) \leq \varepsilon + \delta \right\} \geq R_0 - \gamma > R + 2\gamma. \quad (59)$$

Since $F_w(R|P_X^{(\delta)})$ is a non-decreasing function of R , (59) implies

$$F_w(R + 2\gamma|P_X^{(\delta)}) \leq \varepsilon + \delta. \quad (60)$$

On the other hand, by setting $M_n = e^{nR}$, (3) holds trivially. We now consider the ensemble of random codes for which n symbols of each codeword are randomly chosen according to $P_X^{(\delta)}$ i.i.d. That is, $P_{X^n}(x) = \prod_{i=1}^n P_X^{(\delta)}(x_i)$ ($\forall x \in \mathcal{X}^n$). Then Lemma 5 guarantees that there exists an (n, M_n, ε_n) code satisfying

$$\begin{aligned} \varepsilon_n & \leq \sum_{\ell \in \Omega} w_\ell \Pr \left\{ \frac{1}{n} \log \frac{W_\ell^n(Y_\ell^n|X^n)}{(P_X^{(\delta)} W_\ell)^{\times n}(Y_\ell^n)} \leq R \right. \\ & \quad \left. + \gamma + \frac{1}{n} \log \frac{1}{w_\ell} \right\} + e^{-n\gamma}. \end{aligned} \quad (61)$$

Taking the limes superior with respect n on both sides in (61),

$$\begin{aligned} \limsup_{n \rightarrow \infty} \varepsilon_n & \leq \sum_{\ell \in \Omega} w_\ell \limsup_{n \rightarrow \infty} \Pr \left\{ \frac{1}{n} \log \frac{W_\ell^n(Y_\ell^n|X^n)}{(P_X^{(\delta)} W_\ell)^{\times n}(Y_\ell^n)} \right. \\ & \quad \left. \leq R + \gamma + \frac{1}{n} \log \frac{1}{w_\ell} \right\} \\ & \leq \sum_{\ell \in \Omega} w_\ell \mathbf{1} \left\{ I_{P_X^{(\delta)}}(X; Y_\ell) \leq R + 2\gamma \right\} \end{aligned} \quad (62)$$

$$= F_w(R + 2\gamma|P_X^{(\delta)}) \leq \varepsilon + \delta \quad (63)$$

holds by the sub-additivity of the limes superior. The inequality in (62) is due to Lemma 7, and the last inequality follows from (60). Since (63) holds for an arbitrary fixed $\delta > 0$,

$$\limsup_{n \rightarrow \infty} \varepsilon_n \leq \varepsilon \quad (64)$$

holds, and thus R is ε -achievable.

APPENDIX A PROOF OF LEMMA 1

A. Property (a): Continuity of $A(P_X, \delta)$ in P_X

Mutual information $I_{P_X}(X; Y_\ell)$ is uniformly continuous in P_X since the input alphabet \mathcal{X} is finite. Then we have the

following lemma.

Lemma 8: For at most countably many stationary memoryless channels $\{W_\ell\}_{\ell \in \Omega}$, we have

$$\begin{aligned} & \forall \eta > 0, \exists \lambda(\eta) > 0, \forall \ell \in \Omega, \forall P_X, P'_X \in \mathcal{P}(\mathcal{X}) \text{ s.t.} \\ & \|P_X - P'_X\| \leq \lambda(\eta) \Rightarrow |I_{P_X}(X; Y_\ell) - I_{P'_X}(X; Y_\ell)| \leq \eta, \end{aligned} \quad (65)$$

where we define

$$\|P_X - P'_X\| := \sum_{x \in \mathcal{X}} |P_X(x) - P'_X(x)|, \quad (66)$$

the variational distance between P_X and P'_X . \square

Remark 5: This lemma holds for an arbitrary family of uniform continuous functions $\{f_\ell(P_X) \mid f_\ell : \mathcal{D} \rightarrow \mathbb{R}\}$, where \mathcal{D} is a compact set in $\mathcal{P}(\mathcal{X})$. A constant $\lambda(\eta)$ in (65) can be chosen independent of channel index ℓ because of the uniform continuity of $f_\ell(P_X)$. \square

Fix $\eta > 0$ arbitrarily, and choose any $P_X, P'_X \in \mathcal{P}(\mathcal{X})$ satisfying $\|P_X - P'_X\| \leq \lambda(\eta)$. By Lemma 8, we have

$$|I_{P_X}(X; Y_\ell) - I_{P'_X}(X; Y_\ell)| \leq \eta \quad (\forall \ell \in \Omega). \quad (67)$$

Since (67) implies

$$\sum_{\ell \in \Omega} w_\ell \mathbf{1} \{I_{P'_X}(X; Y_\ell) \leq R\} \geq \sum_{\ell \in \Omega} w_\ell \mathbf{1} \{I_{P_X}(X; Y_\ell) \leq R - \eta\},$$

we have a chain of expansions

$$\begin{aligned} A(P'_X, \delta) & \leq \sup \left\{ R \mid \sum_{\ell \in \Omega} w_\ell \mathbf{1} \{I_{P_X}(X; Y_\ell) \leq R - \eta\} \leq \delta \right\} \\ & = \sup \left\{ R + \eta \mid \sum_{\ell \in \Omega} w_\ell \mathbf{1} \{I_{P_X}(X; Y_\ell) \leq R\} \leq \delta \right\} \\ & = A(P_X, \delta) + \eta. \end{aligned} \quad (68)$$

By the same argument, we also have

$$A(P_X, \delta) \leq A(P'_X, \delta) + \eta. \quad (69)$$

Since P_X, P'_X are arbitrarily chosen, (68) and (69) imply

$$|A(P_X, \delta) - A(P'_X, \delta)| \leq \eta, \quad (70)$$

and thus the function $A(P_X, \delta)$ is continuous in P_X .

B. Property (d): Right Continuity of $\tilde{R}(\delta|\mathcal{D})$ in δ

The function $\tilde{R}(\cdot|\mathcal{D})$ is non-decreasing in δ because of Property (b) of $A(P_X, \cdot)$. Then it is sufficient to show

$$\lim_{k \rightarrow \infty} \tilde{R}(\delta + \lambda_k|\mathcal{D}) = \tilde{R}(\delta|\mathcal{D}) \quad (71)$$

by fixing $\delta \in [0, 1)$ and a decreasing sequence $\{\lambda_k > 0 \mid \lambda_1 > \lambda_2 > \dots \rightarrow 0\}$ arbitrarily. We denote by \mathbb{N} the set of all natural numbers. We assign an index $k \in \mathbb{N}$ to $A(P_X, \delta + \lambda_k)$ and relabel as $\tilde{A}_k(P_X|\delta) := A(P_X, \delta + \lambda_k)$.

By the properties of $A(P_X, \delta)$ (Property (a)–(c)), we have the following:

- (i) $\{\tilde{A}_k(P_X|\delta)\}_{k \in \mathbb{N}}$ is a monotonically decreasing sequence of functions in k .
- (ii) $\lim_{k \rightarrow \infty} \tilde{A}_k(P_X|\delta) = A(P_X, \delta)$ (pointwise convergence in P_X).
- (iii) $A(P_X, \delta)$ is a continuous function of P_X .

Thus, since a monotonically decreasing sequence of functions converges pointwise to a continuous function over a compact set \mathcal{D} , Dini's theorem holds, and $\{\tilde{A}_k(P_X|\delta)\}_{k \in \mathbb{N}}$ converge to $A(P_X, \delta)$ uniformly. By the uniform convergence, we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \max_{P_X \in \mathcal{D}} \tilde{A}_k(P_X|\delta) &= \max_{P_X \in \mathcal{D}} \lim_{k \rightarrow \infty} \tilde{A}_k(P_X|\delta) \\ &= \max_{P_X \in \mathcal{D}} A(P_X, \delta) \end{aligned} \quad (72)$$

(c.f. [1, Lemma 2]). By the relation

$$\tilde{R}(\delta + \lambda_k|\mathcal{D}) = \max_{P_X \in \mathcal{D}} \tilde{A}_k(P_X|\delta) \quad (73)$$

and the definition of $\tilde{R}(\delta|\mathcal{D})$, (72) means (71).

APPENDIX B PROOF OF LEMMA 2

Fix an input probability distribution $P_X \in \mathcal{P}(\mathcal{X})$ arbitrarily. It is easily verified that the l.h.s. of (17) can be expressed as

$$\begin{aligned} &\sup \{R | F_w(R|P_X) \leq \varepsilon\} \\ &= \sup \left\{ R \mid \sum_{\ell} w_{\ell} \mathbf{1}\{I_{P_X}(X; Y_{\ell}) < R\} \leq \varepsilon \right\} \\ &= \sup \left\{ R \mid \sum_{\ell} w_{\ell} \mathbf{1}\{I_{P_X}(X; Y_{\ell}) \geq R\} \geq 1 - \varepsilon \right\}. \end{aligned} \quad (74)$$

Therefore, defining

$$A(\varepsilon|P_X) := \sup \left\{ R \mid \sum_{\ell} w_{\ell} \mathbf{1}\{I_{P_X}(X; Y_{\ell}) \geq R\} \geq 1 - \varepsilon \right\}, \quad (75)$$

$$B(\varepsilon|P_X) := \sup_{\{S \subseteq \Omega | w(S) \geq 1 - \varepsilon\}} \inf_{\ell \in S} I_{P_X}(X; Y_{\ell}), \quad (76)$$

we shall show $A(\varepsilon|P_X) = B(\varepsilon|P_X)$.

(i) Proof of $A(\varepsilon|P_X) \geq B(\varepsilon|P_X)$:

Set $R_0 := B(\varepsilon|P_X)$. By the definition of $B(\varepsilon|P_X)$, for any fixed $\gamma > 0$, there exists $S_0 \subseteq \Omega$ satisfying $w(S_0) \geq 1 - \varepsilon$ and

$$R_0 \leq \inf_{k \in S_0} I_{P_X}(X; Y_k) + \gamma. \quad (77)$$

Also, by the definition of infimum, we have a chain of inequalities

$$\begin{aligned} &\inf_{k \in S_0} I_{P_X}(X; Y_k) \\ &= \sup \left\{ R \mid I_{P_X}(X; Y_{\ell}) \geq R \ (\forall \ell \in S_0) \right\} \\ &= \sup \left\{ R \mid I_{P_X}(X; Y_{\ell}) \geq R \ (\forall \ell \in S_0), \right. \\ &\quad \left. \sum_{\ell \in S_0} w_{\ell} \mathbf{1}\{I_{P_X}(X; Y_{\ell}) \geq R\} \geq 1 - \varepsilon \right\} \\ &\leq \sup \left\{ R \mid \sum_{\ell \in S_0} w_{\ell} \mathbf{1}\{I_{P_X}(X; Y_{\ell}) \geq R\} \geq 1 - \varepsilon \right\} \\ &= A(\varepsilon|P_X). \end{aligned} \quad (78)$$

By (77) and (78), we have

$$R_0 - \gamma \leq A(\varepsilon|P_X), \quad (79)$$

concluding $R_0 \leq A(\varepsilon|P_X)$ since $\gamma > 0$ is fixed arbitrarily,

(ii) Proof of $A(\varepsilon|P_X) \leq B(\varepsilon|P_X)$:

We define the set

$$S(\rho) := \{\ell \in \Omega | I_{P_X}(X; Y_{\ell}) \geq \rho\} \quad (80)$$

for $\rho > 0$. It should be noticed that

$$w(S(\rho_1)) \geq w(S(\rho_2)) \quad (81)$$

for any $0 < \rho_1 \leq \rho_2$.

Consider the value $\rho^* > 0$ satisfying the following conditions:

$$w(S(\rho^* - \gamma)) \geq 1 - \varepsilon \quad (\forall \gamma > 0), \quad (82)$$

$$w(S(\rho)) < 1 - \varepsilon \quad (\forall \rho > \rho^*). \quad (83)$$

For an arbitrarily fixed $\eta > 0$, we have $S(\rho^* + \eta) \subset S(\rho^* - \eta)$ and

$$\sum_{\ell \in \Omega} w_{\ell} \mathbf{1}\{I_{P_X}(X; Y_{\ell}) \geq \rho^* + \eta\} = w(S(\rho^* + \eta)) < 1 - \varepsilon \quad (84)$$

from (83). Since every $R > 0$ such that

$$\sum_{\ell \in \Omega} w_{\ell} \mathbf{1}\{I_{P_X}(X; Y_{\ell}) \geq R\} < 1 - \varepsilon \quad (85)$$

satisfies $R \geq A(\varepsilon|P_X)$ by the definition of $A(\varepsilon|P_X)$, (84) implies

$$A(\varepsilon|P_X) \leq \rho^* + \eta. \quad (86)$$

Meanwhile, we have

$$\rho^* - \eta \leq \inf_{k \in S(\rho^* - \eta)} I_{P_X}(X; Y_k) \leq B(\varepsilon|P_X), \quad (87)$$

where the first inequality follows from the definition of $S(\rho)$, and the second one follows from the fact $w(S(\rho^* - \eta)) \geq 1 - \varepsilon$ and the definition of $B(\varepsilon|P_X)$. It follows from (86) and (87) that

$$A(\varepsilon|P_X) \leq B(\varepsilon|P_X) + 2\eta \quad (88)$$

holds. Since $\eta > 0$ is arbitrarily fixed, it concludes $A(\varepsilon|P_X) \leq B(\varepsilon|P_X)$.

APPENDIX C PROOF OF LEMMA 3

Suppose that the decoder $\xi : \mathcal{Y}^n \rightarrow \{1, \dots, M_n\}$ attains the error probability ε_n without loss of generality. Setting $\xi_{\ell} = \xi$ ($\forall \ell \in \Omega$), and denoting by ξ_{ℓ}^{ML} the maximum likelihood decoder over W_{ℓ}^n , we have

$$\begin{aligned} \varepsilon_n &= 1 - \frac{1}{M_n} \sum_{i=1}^{M_n} \sum_{\mathbf{y} \in \mathcal{Y}^n} W^n(\mathbf{y} | \phi(i)) \xi(i | \mathbf{y}) \\ &= \sum_{\ell \in \Omega} w_{\ell} \left\{ 1 - \frac{1}{M_n} \sum_{i=1}^{M_n} \sum_{\mathbf{y} \in \mathcal{Y}^n} W_{\ell}^n(\mathbf{y} | \phi(i)) \xi_{\ell}(i | \mathbf{y}) \right\} \\ &\geq \sum_{\ell \in \Omega} w_{\ell} \left\{ 1 - \frac{1}{M_n} \sum_{i=1}^{M_n} \sum_{\mathbf{y} \in \mathcal{Y}^n} W_{\ell}^n(\mathbf{y} | \phi(i)) \xi_{\ell}^{\text{ML}}(i | \mathbf{y}) \right\}. \end{aligned} \quad (89)$$

Here, the terms inside the brace $\{\cdot\}$ in (89) corresponds to

the average error probability $\varepsilon_n^{(\ell)}$ of the decoder $\xi_\ell = \xi$ over W_ℓ^n , and the terms inside the brace $\{\cdot\}$ in (90) denotes the average error probability $\varepsilon_\ell^{\text{ML}}$ of the maximum likelihood decoder ξ_ℓ^{ML} . The inequality in (90) follows from the fact that the maximum likelihood decoder attains the minimum error probability among all decoders over W_ℓ^n . The probability $\varepsilon_\ell^{\text{ML}}$ can be evaluated by using $\alpha_\beta(\cdot, \cdot)$ according to the following lemma shown by Vazquez-Vilar et al. [13].

Lemma 9 (Vazquez-Vilar et al. [13]): For a given code \mathcal{C}_n of length n and the number of codewords M_n , the average error probability of the maximum likelihood decoder over the channel W_ℓ^n is given by

$$\varepsilon_\ell^{\text{ML}} = \sup_{Q_{Y_\ell^n}} \alpha_{\frac{1}{M_n}}(P_{X^n} W_\ell^n, P_{X^n} Q_{Y_\ell^n}). \quad (91)$$

Here, P_{X^n} denotes the uniform distribution on \mathcal{C}_n , and the max on the r.h.s. is taken over all probability measures on \mathcal{Y}^n . \square

Applying Lemma 9 for (90) yields

$$\varepsilon_n \geq \sum_{\ell \in \Omega} w_\ell \sup_{Q_{Y_\ell^n}} \alpha_{\frac{1}{M_n}}(P_{X^n} W_\ell^n, P_{X^n} Q_{Y_\ell^n}). \quad (92)$$

Thus, (25) holds.

By using a duality of (α, β_α) and (α_β, β) in simple hypothesis testing, (91) implies

$$\frac{1}{M_n} \geq \beta_{\varepsilon_\ell^{\text{ML}}}(P_{X^n} W_\ell^n, P_{X^n} Q_{Y_\ell^n}) \quad (93)$$

for every fixed $Q_{Y_\ell^n}$, which can be easily verified by considering the region of possible pairs of (α, β) (c.f. [8, Figure 3.1]). Since $\varepsilon_\ell^{\text{ML}} \leq \varepsilon_n^{(\ell)}$, we have

$$\beta_{\varepsilon_\ell^{\text{ML}}}(P_{X^n} W_\ell^n, P_{X^n} Q_{Y_\ell^n}) \geq \beta_{\varepsilon_n^{(\ell)}}(P_{X^n} W_\ell^n, P_{X^n} Q_{Y_\ell^n}) \quad (94)$$

for any given $Q_{Y_\ell^n}$, yielding the inequality

$$\frac{1}{M_n} \geq \sup_{Q_{Y_\ell^n}} \beta_{\varepsilon_n^{(\ell)}}(P_{X^n} W_\ell^n, P_{X^n} Q_{Y_\ell^n}), \quad (95)$$

from (93). Lower bounding the r.h.s. of (95) by fixing some $\{Q_{Y_\ell^n}\}_{\ell \in \Omega}$ and taking the mixture with the mixing ratio $\{w_\ell\}_{\ell \in \Omega}$ conclude that (26) holds.

APPENDIX D PROOF OF LEMMA 4

We first set $R_0 := -\log \sum_{\ell \in \Omega} w_\ell \beta_{\varepsilon_\ell}(P_\ell, Q_\ell) + \log \delta$ and denote by ξ_ℓ^* a probabilistic test that attains $\beta_{\varepsilon_\ell}(P_\ell, Q_\ell)$ in the hypothesis testing (28). We denote by $T^* \in \{H_0, H_1\}$ the random variable corresponding to the hypothesis estimated by this test. That is, $P_\ell\{T^* = H_1\} = \varepsilon_\ell$ and $Q_\ell\{T^* = H_0\} = \beta_{\varepsilon_\ell}(P_\ell, Q_\ell)$ hold by the well-known Neyman-Pearson lemma.

Then a standard bounding technique gives

$$\begin{aligned} 1 - \varepsilon_\ell &= P_\ell\{T^* = H_0\} \\ &= P_\ell\left\{T^* = H_0, \log \frac{P_\ell(Z_\ell)}{Q_\ell(Z_\ell)} > R_0\right\} \\ &\quad + P_\ell\left\{T^* = H_0, \log \frac{P_\ell(Z_\ell)}{Q_\ell(Z_\ell)} \leq R_0\right\} \\ &\leq P_\ell\left\{\log \frac{P_\ell(Z_\ell)}{Q_\ell(Z_\ell)} > R_0\right\} + e^{R_0} Q_\ell\{T^* = H_0\}, \end{aligned} \quad (96)$$

and this implies

$$\varepsilon_\ell \geq P_\ell\left\{\log \frac{P_\ell(Z_\ell)}{Q_\ell(Z_\ell)} \leq R_0\right\} - e^{R_0} \beta_{\varepsilon_\ell}(P_\ell, Q_\ell) \quad (\forall \ell \in \Omega) \quad (97)$$

by the definition of ξ_ℓ^* . Since $\{\varepsilon_\ell\}_{\ell \in \Omega}$ satisfies $\sum_{\ell \in \Omega} w_\ell \varepsilon_\ell = \varepsilon$, taking the mixture of both sides with $\{w_\ell\}_{\ell \in \Omega}$ yields

$$\begin{aligned} \varepsilon &\geq \sum_{\ell \in \Omega} w_\ell P_\ell\left\{\log \frac{P_\ell(Z_\ell)}{Q_\ell(Z_\ell)} \leq R_0\right\} - e^{R_0} \sum_{\ell \in \Omega} w_\ell \beta_{\varepsilon_\ell}(P_\ell, Q_\ell) \\ &= \sum_{\ell \in \Omega} w_\ell P_\ell\left\{\log \frac{P_\ell(Z_\ell)}{Q_\ell(Z_\ell)} \leq R_0\right\} - \delta. \end{aligned} \quad (98)$$

Here, the equality simply follows from the definition of R_0 . (98) indicates

$$\begin{aligned} R_0 &\leq \sup \left\{ R \mid \sum_{\ell \in \Omega} w_\ell P_\ell\left\{\log \frac{P_\ell(Z_\ell)}{Q_\ell(Z_\ell)} \leq R\right\} \leq \varepsilon + \delta \right\} \\ &= D_s^{\varepsilon+\delta}(\{P_\ell\} \parallel \{Q_\ell\}), \end{aligned} \quad (99)$$

and thus (29) holds.

APPENDIX E PROOF OF LEMMA 7

Fix $\gamma > 0$ and $\{\rho_{\ell,n} \geq 0\}$ arbitrarily. We define

$$\begin{aligned} B_\ell^{(n)}(\gamma) &:= \left\{ (\mathbf{x}, \mathbf{y}) \in \mathcal{X}^n \times \mathcal{Y}^n \mid \right. \\ &\quad \left. \left| \frac{1}{n} \log \frac{W_\ell^n(\mathbf{y}|\mathbf{x})}{(P_X W_\ell)^{\times n}(\mathbf{y})} - I_{P_X}(X; Y_\ell) \right| \leq \gamma \right\} \end{aligned} \quad (100)$$

and use a standard bounding technique for each $\ell \in \Omega$ to expand

$$\begin{aligned} &\Pr \left\{ \frac{1}{n} \log \frac{W_\ell^n(Y_\ell^n|X^n)}{(P_X W_\ell)^{\times n}(Y_\ell^n)} \leq R + \rho_{\ell,n} \right\} \\ &\leq \Pr \left\{ \frac{1}{n} \log \frac{W_\ell^n(Y_\ell^n|X^n)}{(P_X W_\ell)^{\times n}(Y_\ell^n)} \leq R + \rho_{\ell,n}, \right. \\ &\quad \left. (X^n, Y_\ell^n) \in B_\ell^{(n)}(\gamma) \right\} + \Pr \left\{ (X^n, Y_\ell^n) \notin B_\ell^{(n)}(\gamma) \right\}. \end{aligned} \quad (101)$$

The random variable $\log \frac{W_\ell^n(Y_\ell^n|X^n)}{(P_X W_\ell)^{\times n}(Y_\ell^n)}$ is a sum of independent random variables. Then, similar to Lemma 6, we can apply the Chebyshev inequality to the second term of (101) and obtain

$$\Pr \left\{ (X^n, Y_\ell^n) \notin B_\ell^{(n)}(\gamma) \right\} \leq \frac{A(\gamma)}{n} \quad (102)$$

with some constant $A(\gamma) \geq 0$. It should be noticed that the variance of the random variable $\frac{1}{n} \log \frac{W_\ell^n(Y_\ell^n|X^n)}{(P_X W_\ell)^{\times n}(Y_\ell^n)}$ is

uniformly bounded in ℓ because \mathcal{X} is finite (c.f. [5, Remark 3.1.1]), and thus a constant $A(\gamma)$ can be chosen independently of ℓ . On the other hand, the first term of (101) can be bounded as

$$\Pr \left\{ \frac{1}{n} \log \frac{W_\ell^n(Y_\ell^n | X^n)}{(P_X W_\ell)^{\times n}(Y_\ell^n)} \leq R + \rho_{\ell,n}, (X^n, Y_\ell^n) \in B_\ell^{(n)}(\gamma) \right\} \leq \mathbf{1} \{I_{P_X}(X; Y_\ell) - \gamma \leq R + \rho_{\ell,n}\}, \quad (103)$$

which can be verified as follows: (i) If $I_{P_X}(X; Y_\ell) - \gamma \leq R + \rho_{\ell,n}$, (103) holds trivially because $\mathbf{1} \{I_{P_X}(X; Y_\ell) - \gamma \leq R + \rho_{\ell,n}\} = 1$, and (ii) If $I_{P_X}(X; Y_\ell) - \gamma > R + \rho_{\ell,n}$, we have

$$\Pr \left\{ \frac{1}{n} \log \frac{W_\ell^n(Y_\ell^n | X^n)}{(P_X W_\ell)^{\times n}(Y_\ell^n)} \leq R + \rho_{\ell,n}, (X^n, Y_\ell^n) \in B_\ell^{(n)}(\gamma) \right\} = 0 \quad (104)$$

because

$$I(X; Y_\ell) - \gamma \leq \frac{1}{n} \log \frac{W_\ell^n(\mathbf{y} | \mathbf{x})}{(P_X W_\ell)^{\times n}(\mathbf{y})}$$

for all $(\mathbf{x}, \mathbf{y}) \in B_\ell^{(n)}(\gamma)$. This implies that (103) also holds.

By (101)–(103), we obtain

$$\Pr \left\{ \frac{1}{n} \log \frac{W_\ell^n(Y_\ell^n | X^n)}{(P_X W_\ell)^{\times n}(Y_\ell^n)} \leq R + \rho_{\ell,n} \right\} \leq \mathbf{1} \{I_{P_X}(X; Y_\ell) \leq R + \rho_{\ell,n} + \gamma\} + \frac{A(\gamma)}{n}. \quad (105)$$

Taking the limes superior with respect n on both sides yields

$$\limsup_{n \rightarrow \infty} \Pr \left\{ \frac{1}{n} \log \frac{W_\ell^n(Y_\ell^n | X^n)}{(P_X W_\ell)^{\times n}(Y_\ell^n)} \leq R + \rho_{\ell,n} \right\} \leq \mathbf{1} \{I_{P_X}(X; Y_\ell) \leq R + 2\gamma\}, \quad (106)$$

concluding (55) since $\gamma > 0$ is arbitrary.

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